

Thermal field theory of the Tsallis statistics

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Motivation

- Boltzmann-Gibbs (BG) statistical mechanics may not be always sufficient to study a system having fluctuations (of temperature, number density etc.) inside their boundary, and/or they may experience long-range correlation.
- Hadronic transverse momentum distribution is describable by the "Tsallis-like" distributions up to a very high range of transverse momentum.

Introduction

The Tsallis statistics begins with a generalized definition of entropy:

$$S_q [\hat{\rho}] = \frac{1 - \text{Tr} \hat{\rho}^q}{1 - q}, \text{ with } \text{Tr} \hat{\rho} = 1 \quad \text{Physica A 261 (1998) 534}$$

The thermal expectation value given by $\langle \hat{A} \rangle_q = \text{Tr}(\{\hat{\rho}\}^q \hat{A})$.

S_q approaches the Boltzmann-Gibbs-Shannon entropy given by

$S = -\text{Tr} \hat{\rho} \ln \hat{\rho}$ when q approaches 1.

By extremizing the potential function (for chemical potential, $\mu = 0$),

$$\Omega = \langle \hat{H} \rangle - TS_q,$$

for temperature $T = \beta^{-1}$ with respect to density matrix, we get,

Density matrix in quantum Tsallis statistics

$$\hat{\rho} = Z_q^{-1} \left[1 + (q - 1)\beta \hat{H} \right]^{\frac{1}{1-q}}.$$

Tsallis propagator: definition

Tsallis thermal propagator for the real scalar field $\phi(x = \tau, \vec{x})$ on a contour C can be defined as: **Hadrons at Finite Temperature, S. Mallik, and S. Sarkar**

$$\begin{aligned} D(x, x') &= i\langle \hat{T}_c \phi(x) \phi(x') \rangle_q \\ &= i\theta_c(\tau - \tau') \langle \phi(x) \phi(x') \rangle_q + i\theta_c(\tau' - \tau) \langle \phi(x') \phi(x) \rangle_q \\ &= \theta_c(\tau - \tau') D_+(x, x') + \theta_c(\tau' - \tau) D_-(x, x'), \end{aligned}$$

where, $D_+(x, x') = i\langle \phi(x) \phi(x') \rangle_q$ and $D_-(x, x') = i\langle \phi(x') \phi(x) \rangle_q$

Using $\sum_m |m\rangle \langle m| = \mathbf{1}$, replacing the Heisenberg field $\phi(x)$ by Schrodinger field as $\phi(x) = e^{i\hat{H}\tau} \phi(0, \vec{x}) e^{-i\hat{H}\tau}$ and $\hat{H}|m\rangle = E_m|m\rangle$, we may write,

$$D_+(x, x') = iZ_q^{-1} \sum_{m,n} e^{iE_m \left(\tau - \tau' + \frac{i \ln[1 + (q-1)\beta E_m]}{(q-1)E_m} \right)} e^{-iE_n(\tau - \tau')} \langle m | \phi(0, \vec{x}) | n \rangle \langle n | \phi(0, \vec{x}') | m \rangle,$$

Convergence condition

The sum over m and n converge respectively for,

$$\operatorname{Im} \left[\tau - \tau' + i \frac{\ln[1 + (q-1)\beta E_m]}{(q-1)E_m} \right] \geq 0 \quad \text{and} \quad \operatorname{Im}(\tau - \tau') \leq 0$$

Combining these two we get,

$$-\frac{\ln[1 + (q-1)\beta E_m]}{(q-1)E_m} \leq \operatorname{Im}(\tau - \tau') \leq 0$$

In the limit $q \rightarrow 1$, the the above convergence condition becomes that of the Boltzmann-Gibbs statistics,

$$-\beta \leq \operatorname{Im}(\tau - \tau') \leq 0.$$

KMS relation in the Tsallis statistics

Kubo-Martin-Schwinger (KMS) relation:

$$\begin{aligned} D_-(x, x') &= i\langle \phi(x')\phi(x) \rangle_q \\ &= iZ_q^{-1} \text{Tr}' \left[\left\{ e_q^{-\beta \hat{H}} \right\}^q \phi(\tau', \vec{x}') \phi(\tau, \vec{x}) \right] \\ &= D_+ \left(\vec{x}, \vec{x}', \tau - \frac{iq}{E_m(q-1)} \ln [1 + (q-1)\beta E_m], \tau' \right) \\ D_-(x, x') &= D_+ \left(\vec{x}, \vec{x}', \tau - i\bar{\beta}, \tau' \right), \end{aligned}$$

where

$$\bar{\beta} = \frac{q}{(q-1)E_m} \ln [1 + (q-1)\beta E_m],$$

In the limit $q \rightarrow 1$, the above reduces to its Boltzmann-Gibbs counterpart:

$$D_-(x, x') = D_+ \left(\vec{x}, \vec{x}', \tau - i\beta, \tau' \right),$$

Deriving Tsallis propagator using the differential equation method

$$(\square_c + m^2)D(x, x') = \delta_c^4(x - x').$$

Taking Fourier transform, $\left(\frac{\partial^2}{\partial \tau^2} + \omega_{\mathbf{k}}^2 \right) D(\vec{k}, \tau, \tau') = \delta_c(\tau - \tau')$,

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$ is single particle energy, $\mathbf{k} \equiv |\vec{k}|$. For $\tau \neq \tau'$,

$$\begin{aligned} D_+(\vec{k}, \tau, \tau') &= A_1(\tau') e^{-i\omega_{\mathbf{k}}\tau} + A_2(\tau') e^{i\omega_{\mathbf{k}}\tau} \quad (\tau > \tau'), \\ D_-(\vec{k}, \tau, \tau') &= B_1(\tau') e^{-i\omega_{\mathbf{k}}\tau} + B_2(\tau') e^{i\omega_{\mathbf{k}}\tau} \quad (\tau < \tau'). \end{aligned}$$

$$\begin{aligned} D(\vec{k}, \tau_i, \tau'_j) &= \frac{i}{2\omega_{\mathbf{k}}} \left[\theta_c(\tau_i - \tau'_j) \left\{ (1 + n_T) e^{-i\omega_{\mathbf{k}}(\tau_i - \tau'_j)} + n_T e^{i\omega_{\mathbf{k}}(\tau_i - \tau'_j)} \right\} \right. \\ &\quad \left. + \theta_c(\tau'_j - \tau_i) \left\{ n_T e^{-i\omega_{\mathbf{k}}(\tau_i - \tau'_j)} + (1 + n_T) e^{i\omega_{\mathbf{k}}(\tau_i - \tau'_j)} \right\} \right] \end{aligned}$$

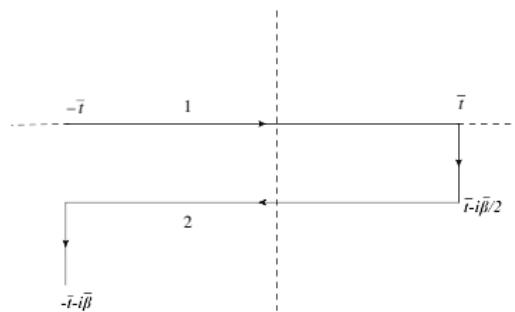
Where the distribution in the Tsallis statistics:

$$n_T(\omega_{\mathbf{k}}) = \frac{1}{[1 + (q - 1)\beta\omega_{\mathbf{k}}]^{\frac{q}{q-1}} - 1},$$

Tsallis propagator

Temporal Fourier transform

$$D_{ij}(\vec{k}, k_0) = \int_{-\infty}^{\infty} dt e^{ik_0(t-t')} D(\vec{k}, \tau_i, \tau_j').$$



We can take two points on two horizontal lines in four different ways,

- τ and τ' both lie on line 1, then $\theta_c(\tau_1 - \tau'_1) = \theta(t - t')$.
- τ and τ' both lie on line 2, then $\theta_c(\tau_1 - \tau'_2) = \theta(t' - t)$.
- τ lies on line 1 and τ' lies on line 2, then $\theta_c(\tau_2 - \tau'_1) = 1$.
- τ lies on line 2 and τ' lies on line 1, then $\theta_c(\tau_2 - \tau'_2) = 0$.

Tsallis propagator: components

Components of the real time thermal propagator:

$$\begin{aligned} D_{11}(\vec{k}, k_0) &= \frac{i}{2\omega_k} \int_{-\infty}^{\infty} dt e^{ik_0(t-t')} \left[\theta(t-t') \left\{ (1+n_T)e^{-i\omega_k(t-t')} + n_T e^{i\omega_k(t-t')} \right\} \right. \\ &\quad \left. \theta(t'-t) \left\{ n_T e^{-i\omega_k(t-t')} + (1+n_T)e^{i\omega_k(t-t')} \right\} \right] \\ &= \frac{-1}{k^2 - m^2 + i\epsilon} + \frac{2\pi i \delta(k^2 - m^2)}{[1 + (q-1)\beta\omega_k]^{\frac{q}{q-1}} - 1}, \end{aligned}$$

$$D_{12}(\vec{k}, k_0) = 2\pi i \sqrt{n_T(1+n_T)} \delta(k^2 - m^2)$$

$$D_{21}(\vec{k}, k_0) = D_{12}(\vec{k}, k_0) \quad \text{and} \quad D_{22}(\vec{k}, k_0) = -D_{11}^*(\vec{k}, k_0)$$

In the limit $q \rightarrow 1$, we get back the real time BG bosonic propagator,

$$\begin{aligned} D_{11}^{\text{BG}}(\vec{k}, k_0) &= \frac{-1}{k^2 - m^2 + i\epsilon} + \frac{2\pi i \delta(k^2 - m^2)}{e^{\beta\omega_k} - 1} \\ D_{12}^{\text{BG}}(\vec{k}, k_0) &= 2\pi i \sqrt{n(1+n)} \delta(k^2 - m^2), \end{aligned}$$

Free Tsallis thermal propagator for the fermions

The Dirac thermal propagator in the Tsallis statistics may be defined by,

$$\begin{aligned} S(x, x') &\equiv i\langle T_c \psi(x) \bar{\psi}(x') \rangle_q \\ &= iZ_q^{-1} \text{Tr}[\rho^q T_c \psi(x) \bar{\psi}(x')]. \end{aligned}$$

The matrix representing the real time propagator $\mathbf{S}(\vec{p}, p_0)$ is given by,

$$\mathbf{S}(\vec{p}, p_0) = (\not{p} + m) \begin{pmatrix} \Delta_F(p) - 2\pi i N_1^2 \delta(p^2 + m^2) & -2\pi i N_1 N_2 \delta(p^2 + m^2) \\ 2\pi i N_1 N_2 \delta(p^2 + m^2) & -\Delta_F^*(p) + 2\pi i N_1^2 \delta(p^2 + m^2) \end{pmatrix}.$$

where, Tsallis FD distribution function: $f_T(\omega) \equiv \frac{1}{[1 + (q-1)\beta\omega]^{\frac{q}{q-1}} + 1}$,

$$N_1 \equiv \sqrt{f_T} [\theta(p_0) + \theta(-p_0)] \quad \text{and} \quad N_2 \equiv \sqrt{(1 - f_T)} [\theta(p_0) - \theta(-p_0)].$$

Diagonal form: bosonic and Dirac propagator

The bosonic propagator can be put in a diagonal form,

$$\mathbf{D}(\vec{k}, k_0) = \mathbf{U}(k_0) \begin{pmatrix} \Delta_F & 0 \\ 0 & -\Delta_F^* \end{pmatrix} \mathbf{U}(k_0).$$

The diagonalizing matrix \mathbf{U} is given by,

$$\mathbf{U}(k_0) = \begin{pmatrix} \sqrt{1 + n_T} & \sqrt{n_T} \\ \sqrt{n_T} & \sqrt{1 + n_T} \end{pmatrix},$$

The Dirac propagator can be put in a diagonal form,

$$\mathbf{S}(\vec{p}, p_0) = (\not{p} + m)\mathbf{V} \begin{pmatrix} \Delta_F & 0 \\ 0 & -\Delta_F^* \end{pmatrix} \mathbf{V},$$

where the diagonalizing matrix \mathbf{V} is given by,

$$\mathbf{V} = \begin{pmatrix} N_2 & -N_1 \\ N_1 & N_2 \end{pmatrix}.$$

Diagonalisation of bosonic full propagator

Dyson-Schwinger equation:

$$\mathbf{D}'(k) = \mathbf{D}(k) + \mathbf{D}(k)\boldsymbol{\Pi}(k)\mathbf{D}'(k)$$

The matrix $\mathbf{U}(k_0)$, diagonalizes the free propagator $\mathbf{D}(k_0)$, also diagonalizes the total propagator $\mathbf{D}'(k)$ and the self energy $\boldsymbol{\Pi}(k)$ too,

$$\boldsymbol{\Pi}(k) = \mathbf{U}^{-1}(k) \begin{pmatrix} \bar{\boldsymbol{\Pi}}(k) & 0 \\ 0 & -\bar{\boldsymbol{\Pi}}(k)^* \end{pmatrix} \mathbf{U}^{-1}(k)$$

which in terns diagonalizes the matrix equation to an ordinary equation

$$\bar{D} = \Delta + \Delta \bar{\boldsymbol{\Pi}} \bar{D}$$

having the solution $\bar{D} = \frac{-1}{k^2 - m^2 + \bar{\boldsymbol{\Pi}}}$

with $\text{Re}\bar{\boldsymbol{\Pi}} = \text{Re}\boldsymbol{\Pi}_{11}$, and

$$\text{Im}\bar{\boldsymbol{\Pi}} = \epsilon(k_0) \tanh \left[\frac{q}{2(q-1)} \ln [1 + \beta(q-1)k_0] \right] \text{Im}\boldsymbol{\Pi}_{11}$$

$$\boldsymbol{\Pi}_{22} = -\boldsymbol{\Pi}_{11}^*, \boldsymbol{\Pi}_{21} = \boldsymbol{\Pi}_{12}$$

Diagonalisation of fermionic full propagator

For the Dirac propagator, the Dyson -Schwinger equation reads

$$\mathbf{S}'(p) = \mathbf{S}(p) + \mathbf{S}(p)\boldsymbol{\Sigma}(p)\mathbf{S}'(p)$$

The matrix $\mathbf{V}(p_0)$ diagonalizes the free propagator $\mathbf{S}(p_0)$, also diagonalizes the total propagator $\mathbf{S}'(p)$ and the self energy $\boldsymbol{\Sigma}(p)$.

$$\boldsymbol{\Sigma}(p) = \mathbf{V}^{-1}(p) \begin{pmatrix} \bar{\Sigma}(p) & 0 \\ 0 & -\bar{\Sigma}(p)^* \end{pmatrix} \mathbf{V}^{-1}(p)$$

which reduces the matrix equation to an ordinary equation

$$\bar{S} = S + S\bar{\Sigma}\bar{S}$$

having the solution $\bar{S} = \frac{-1}{\not{p} - m + \bar{\Sigma}}$

with $\text{Re}\bar{\Sigma} = \text{Re}\Sigma_{11}$,

$$\text{Im}\bar{\Sigma} = \epsilon(p_0) \coth \left[\frac{q}{2(q-1)} \ln [1 + \beta(q-1)p_0] \right] \text{Im}\Sigma_{11}.$$

and $\Sigma_{22} = -\Sigma_{11}^*$, $\Sigma_{21} = \Sigma_{12}$

Application: thermal mass shift of the scalar particle

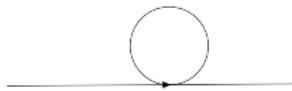


Figure: Leading self energy diagram in ϕ^4 theory.

The Lagrangian density of the ϕ^4 theory is given by,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

The mass shift due to thermal effects, Δm_T^2 can be obtained as

$$\Delta m_T^2 = \frac{\lambda}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\omega_p} \frac{1}{[1 + (q-1)\beta\omega_p]^{\frac{q}{q-1}} - 1}$$

$$\Delta m_T^2 = \frac{\lambda m^2}{16\pi^2} \sum_{s=1}^{\infty} \left\{ \frac{m(q-1)}{T} \right\}^{-\frac{qs}{q-1}} \left[\frac{\Gamma\left(\frac{q(s-2)+2}{2(q-1)}\right) {}_2F_1\left(\frac{q(s-2)+2}{2(q-1)}, \frac{qs}{2(q-1)}; \frac{1}{2}; \frac{T^2}{m^2(q-1)^2}\right)}{\Gamma\left(\frac{sq+q-1}{2(q-1)}\right)} \right. \\ \left. - \frac{T}{m(q-1)} \frac{\Gamma\left(\frac{q(s-1)+1}{2(q-1)}\right) \Gamma\left(\frac{sq+q-1}{q-1}\right) {}_2F_1\left(\frac{q(s-1)+1}{2(q-1)}, \frac{sq+q-1}{2(q-1)}, \frac{3}{2}; \frac{T^2}{m^2(q-1)^2}\right)}{\Gamma\left(\frac{qs}{q-1}\right) \Gamma\left(\frac{q(s+2)-2}{2(q-1)}\right)} \right].$$

for $q \rightarrow 1$, the above integral approaches the Boltzmann-Gibbs result: $\Delta m^2 = \frac{\lambda T^2}{24} + \mathcal{O}(m/T)$.

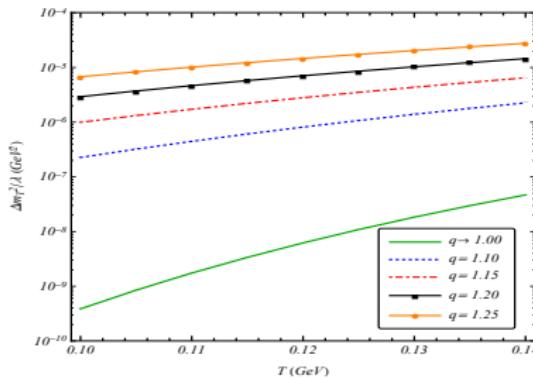


Figure: Variation of the thermal mass of a scalar particle of mass 1.5 GeV with T for the BG (green, solid line) and the Tsallis statistics.

Summary and outlook

- Derived the real time thermal propagators for scalar and Dirac fields within the scope of the Tsallis statistics.
- We observe that, when the entropic parameter q approaches 1, the classical and quantum BG thermal propagators are recovered.
- The thermal mass shift of a scalar particle subjected to ϕ^4 interaction is estimated.
- we observe a enhancement in the scaled mass shift which approaches the Boltzmann-Gibbs limit as q approaches 1.
- This formalism and its extension may help one to compute the quantities like thermal mass, decay rate, energy loss in more realistic situations dealing with fluctuating ambience, and/or long-range correlations.

Collaborators:

**Jan-e Alam
Trambak Bhattacharyya**

Thank you