Latest developments in relativistic hydrodynamics

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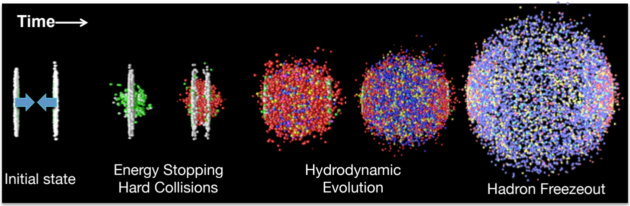
Outline

1.1 Standard model of heavy-ion collisions



T. K. Nayak, Lepton-Photon 2011 Conference

FIRST STAGE — HIGHLY OUT-OF EQUILIBRIUM ($0 < \tau_0 \lesssim 1\,\mathrm{fm}$)

- **initial conditions**, including fluctuations, reflect to large extent the distribution of matter in the colliding nuclei — Glauber model, works by A. Białas and W. Czyż
- **emission of hard probes**: heavy quarks, photons, jets
- **hydrodynamization stage** – the system becomes well described by equations of viscous hydrodynamics — AdS/CFT calculations by R. Janik and his collaborators
- talks by S. Schlichting and R. Venugopalan in this conference

SECOND STAGE — HYDRODYNAMIC EXPANSION ($1\,\mathrm{fm} \lesssim \tau \lesssim 10\,\mathrm{fm}$)

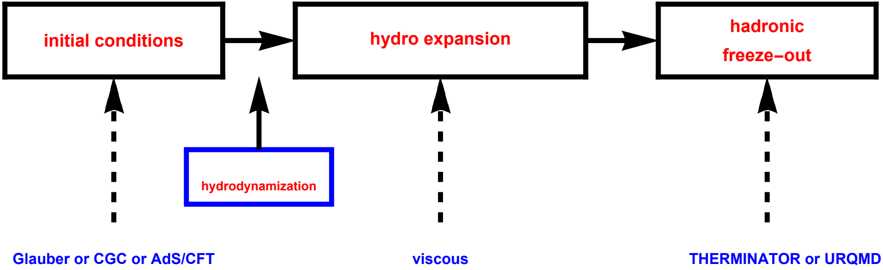
- expansion controlled by viscous hydrodynamics (effective description)
- **thermalization stage**
- **phase transition** from QGP to hadron gas takes place (encoded in the equation of state)
- **equilibrated hadron gas**

THIRD STAGE — FREEZE-OUT

- **freeze-out and free streaming of hadrons** ($10\,\mathrm{fm} \lesssim \tau$)

BARYON NUMBER MOSTLY NEGLECTED IN THIS TALK

STANDARD MODEL (MODULES) of HEAVY–ION COLLISIONS



FLUCTUATIONS IN THE INITIAL STATE / EVENT–BY–EVENT HYDRO / FINAL–STATE FLUCTUATIONS

EQUATION OF STATE = lattice QCD

1 < VISCOSITY < 3 times the lower bound

Danielewicz+Gyulassy (quantum mechanics), Kovtun+Son+Starinets (AdS/CFT)

lower bound on the ratio of shear viscosity to entropy density $\eta/S = 1/(4\pi)$

1.2 From perfect-fluid to viscous hydrodynamics

see also A. Jaiswal's lecture during the students' day

$T(x)$ and $u^\mu(x)$ are fundamental fluid variables

the relativistic perfect-fluid energy-momentum tensor is the most general symmetric tensor which can be expressed in terms of these variables without using derivatives

dynamics of the perfect fluid theory is provided by the conservation equations of the energy-momentum tensor, four equations for the four independent hydrodynamic fields – a self-consistent (hydrodynamic) theory

$$\partial_\mu T^{\mu\nu}_{\rm eq} = 0, \qquad T^{\mu\nu}_{\rm eq} = \mathcal{E} u^\mu u^\nu - \mathcal{P}_{\rm eq}(\mathcal{E})\Delta^{\mu\nu}, \qquad \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \tag{1}$$

$$\mathcal{E}_{\rm eq}(T(x)) = \mathcal{E}(x), \qquad T^{\mu\nu}_{\rm eq}(x) u_\nu(x) = \mathcal{E}(x) u^\mu(x). \tag{2}$$

$$\text{local rest frame:} \qquad u^\mu = (1,0,0,0) \quad \rightarrow \quad T^{\mu\nu}_{\rm eq} = \begin{bmatrix} \mathcal{E} & 0 & 0 & 0 \\ 0 & \mathcal{P}{\rm eq} & 0 & 0 \\ 0 & 0 & \mathcal{P}{\rm eq} & 0 \\ 0 & 0 & 0 & \mathcal{P}{\rm eq} \end{bmatrix} \tag{3}$$

DISSIPATION DOES NOT APPEAR! $u_\nu \partial_\mu T^{\mu\nu}_{\rm eq} = 0 \rightarrow \partial_\mu(\mathcal{S} u^\mu) = 0$

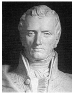
entropy conservation follows from the energy-momentum conservation and the form of the energy-momentum tensor

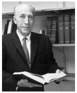
Navier-Stokes hydrodynamics

Claude-Louis Navier, 1785–1836, French engineer and physicist
Sir George Gabriel Stokes, 1819–1903, Irish physicist and mathematician

C. Eckart, Phys. Rev. 58 (1940) 919

L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Pergamon, New York, 1959






complete energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{\rm eq} + \Pi^{\mu\nu} \tag{4}$$

where $\Pi^{\mu\nu} u_\nu = 0$, which corresponds to the Landau definition of the hydrodynamic flow u^μ

$$T^{\mu}_{\ \nu} u^\nu = \mathcal{E}\, u^\mu. \tag{5}$$

It proves useful to further decompose $\Pi^{\mu\nu}$ into two components,

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \Pi \Delta^{\mu\nu}, \tag{6}$$

which introduces the **bulk viscous pressure** Π (the trace part of $\Pi^{\mu\nu}$) and the **shear stress tensor** $\pi^{\mu\nu}$ which is symmetric, $\pi^{\mu\nu} = \pi^{\nu\mu}$, traceless, $\pi^{\mu}_{\ \mu} = 0$, and orthogonal to u^μ, $\pi^{\mu\nu} u_\nu = 0$, $\boxed{\Pi = -\zeta\, \partial_\mu u^\mu, \quad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}}$

Navier-Stokes hydrodynamics

complete energy-momentum tensor

$$\boxed{T^{\mu\nu} = T^{\mu\nu}_{\rm eq} + \pi^{\mu\nu} + \Pi\Delta^{\mu\nu} = T^{\mu\nu}_{\rm eq} + 2\eta\sigma^{\mu\nu} - \zeta\theta\Delta^{\mu\nu}} \tag{7}$$

again four equations for four unknowns

$$\boxed{\partial_\mu T^{\mu\nu} = 0} \tag{8}$$

1) THIS SCHEME DOES NOT WORK IN PRACTICE!

ACAUSAL BEHAVIOR + INSTABILITIES!

2) NEVERTHELESS, THE GRADIENT FORM (7) IS A GOOD APPROXIMATION

FOR SYSTEMS APPROACHING LOCAL EQUILIBRIUM

Great progress has been made in the last years to understand the hydrodynamic gradient expansion by

R. Janik, M. Spaliński , M. P. Heller, P. Witaszczyk and their collaborators

What is the real problem with the relativistic Navier-Stokes theory?

P. Kovtun: Existence of gapped modes (non-hydrodynamic modes) with frequencies that have a positive imaginary part. These are unphysical modes. These (UV) modes are outside of the validity regime of the low-energy hydro approximation.

HYDRODYNAMIC VS. NON-HYDRODYNAMICMODES

perturbations $\sim \exp(-i\omega_k t + ikx)$, hydro modes $\omega_k \to 0$ for $k \to 0$, non-hydro modes $\omega_k \to \text{const} \neq 0$ for $k \to 0$

instability for $\text{Im}(\omega_k) > 0$.

Most popular fix is the Israel-Stewart theory: the hydro equations are coupled to extra UV degrees of freedom, which in turn kill the unphysical UV modes.

Analogy to quantum field theory.

Rayski-Pauli-Villars regularization: introduction of physical heavy particles whose presence regulates the UV behavior.

Israel-Stewart equations

$\Pi, \pi^{\mu\nu}$ promoted to new hydrodynamic variables!

W. Israel and J.M. Stewart, *Transient relativistic thermodynamics and kinetic theory*, Annals of Physics 118 (1979) 341

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\beta_\Pi \theta, \qquad \tau_\Pi \beta_\Pi = \zeta \tag{9}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu}, \qquad \tau_\pi \beta_\pi = 2\eta \tag{10}$$

1) HYDRODYNAMIC EQUATIONS EXPLICITLY DESCRIBE BOTH HYDRODYNAMIC AND NON-HYDRODYNAMIC MODES

2) HYDRODYNAMIC MODES CORRESPOND TO GENUINE HYDRODYNAMIC BEHAVIOR

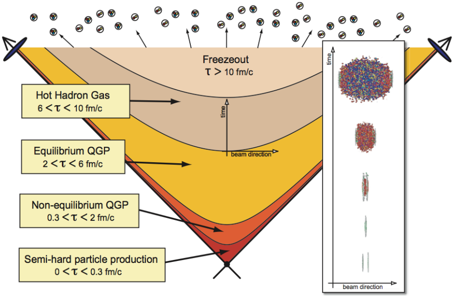
3) NON-HYDRODYNAMIC MODES/TERMS SHOULD BE TREATED AS REGULATORS OF THE THEORY

2 Anisotropic hydrodynamics

2.1 Problems of standard (IS) viscous hydrodynamics

Simplified space-time diagram

space-time diagram for a simplified, one dimensional and boost-invariant expansion



M. Strickland, Acta Phys.Polon. B45 (2014) 2355

evolution governed by the proper time $\tau = \sqrt{t^2 - z^2}$

Pressure anisotropy

space-time gradients in boost-invariant expansion increase the transverse pressure and decrease the longitudinal pressure

$$\mathcal{P}_T = \mathcal{P} + \frac{\pi}{2}, \quad \mathcal{P}_L = \mathcal{P} - \pi, \quad \pi = \frac{4\eta}{3\tau} \tag{11}$$

$$\left(\frac{\mathcal{P}_L}{\mathcal{P}_T}\right)_{\rm NS} = \frac{3\tau T - 16\bar{\eta}}{3\tau T + 8\bar{\eta}}, \quad \bar{\eta} = \frac{\eta}{\mathcal{S}}$$

using the AdS/CFT lower bound for viscosity, $\bar{\eta} = \frac{1}{4\pi}$

RHIC-like initial conditions, $T_0 = 400$ MeV at $\tau_0 = 0.5$ fm/c, $\quad (\mathcal{P}_L/\mathcal{P}_T)_{\rm NS} \approx 0.50$
LHC-like initial conditions, $T_0 = 600$ MeV at $\tau_0 = 0.2$ fm/c, $\quad (\mathcal{P}_L/\mathcal{P}_T)_{\rm NS} \approx 0.35$

problem of early thermalization replaced by the problem of early hydrodynamization

2.2 Concept of aHydro

Thermodynamic & kinetic-theory formulations

Thermodynamic formulation

WF, R. Ryblewski

PRC 83, 034907 (2011), JPG 38 (2011) 015104

1. energy-momentum conservation
   $\partial_\mu T^{\mu\nu} = 0$
2. ansatz for the entropy source, e.g.,
   $\partial(\sigma U^\mu) \propto (\lambda_\perp - \lambda_\parallel)^2/(\lambda_\perp \lambda_\parallel)$

Kinetic-theory formulation

M. Martinez, M. Strickland

NPA 848, 183 (2010), NPA 856, 68 (2011)

1. first moment of the Boltzmann equation = energy-momentum conservation
2. zeroth moment of the Boltzmann equation = specific form of the entropy source

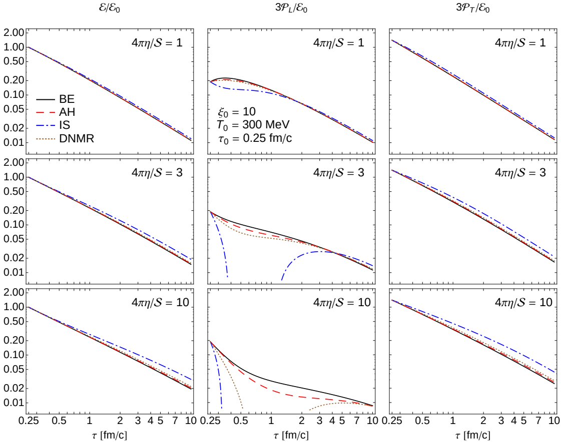
3. Generalized form of the equation of state based on the Romatschke-Strickland (RS) form

generalization of equilibrium/isotropic distributions, frequently used in the studies of anisotropic quark-gluon plasma (here as a modified Boltzmann distribution in the local rest frame)

$$f_{RS} = \exp\left(-\sqrt{\frac{p_\perp^2}{\lambda_\perp^2} + \frac{p_\parallel^2}{\lambda_\parallel^2}}\right) = \exp\left(-\frac{1}{\lambda_\perp}\sqrt{p_\perp^2 + x\, p_\parallel^2}\right) = \exp\left(-\frac{1}{\Lambda}\sqrt{p_\perp^2 + (1+\xi)\, p_\parallel^2}\right)$$

anisotropy parameter $x = 1 + \xi = \left(\frac{\lambda_\perp}{\lambda_\parallel}\right)^2$ and transverse-momentum scale $\lambda_\perp = \Lambda$

WF, R. Ryblewski, M. Strickland, Phys.Rev. C88 (2013) 024903, $m = 0$, boost-invariant, transversally homogeneous system, (0+1) case



aHydro being used and developed now by U. Heinz (Columbus, Ohio), M. Strickland (Kent, Ohio), T. Schaeffer (North Carolina), D. Rischke (Frankfurt), ... ; applied in other branches of physics, cold atoms, ...

3 Hydrodynamic attractors

3.1 Müller-Israel-Stewart attractor

M.P. Heller, M. Spaliński, Phys. Rev. Lett. 115 (2015) 072501

identification of the attractor in the conformal Müller-Israel-Stewart theory (the simplest version of IS)

one-dimensional boost-invariant (Bjorken) expansion, τ - proper time

$$\tau\dot{\varepsilon} = -\frac{4}{3}\varepsilon + \phi, \quad \tau_\pi\dot{\phi} = \frac{4\eta}{3\tau} - \frac{\lambda_1\phi^2}{2\eta^2} - \frac{4\tau_\pi\phi}{3\tau} - \phi \tag{12}$$

(13)

$\varepsilon \sim T^4$, $\zeta = 0$, $\eta = 1/(4\pi)$, $\lambda_1 = \eta/(2\pi T)$, ϕ - the single independent component of the shear stress tensor

$w = \tau T, f = \dot{w}/w$

presence of attractors is possibly connected with early thermalization phenomenon.

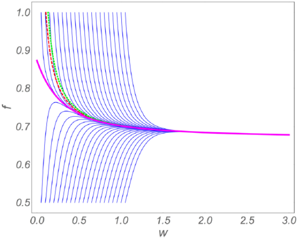
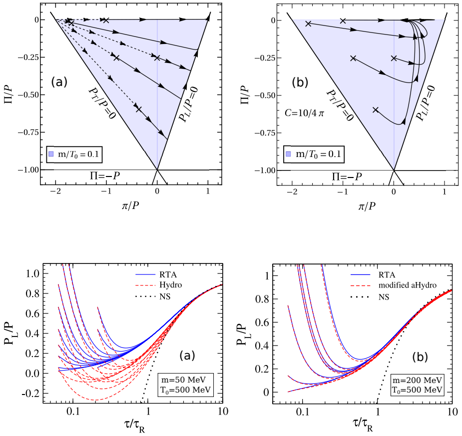


FIG. 1 (color online). The blue lines are numerical solutions of Eq. (8) for various initial conditions; the thick magenta line is the numerically determined attractor. The red dashed and green dotted lines represent first and second order hydrodynamics.

3.2 Non-conformal attractors

S. Jaiswal, S. Pal, C. Chattopadhyay, L. Du, U. Heinz, Acta Phys. Pol. B Proc. Suppl. 16 1-A119 (2023), Quark Matter 2022 Proc.

In kinetic theory, an early-time far-from-equilibrium attractor exists for the scaled longitudinal pressure. Second-order dissipative hydrodynamics fails to accurately describe this attractor, but a modified anisotropic hydrodynamics reproduces it.



more studies of attractors by D. Almaalol et al. (QCD), and S. Plumari et al. (transport theory)

and previous talk by J.-P. Blaizot

4 First-order hydrodynamics revisted

4.1 BDNK theory

Bemfica, Disconzi, Noronha, Kovtun: there is a sensible relativistic hydrodynamics whose only variables are T, u^μ, μ and no extra UV d.o.f. are needed, one needs to choose a suitable out-of-equilibrium definitions of T, u^μ, μ.

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (\mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu) + \mathcal{T}^{\mu\nu} \tag{14}$$

$\mathcal{E}, \mathcal{P}, \mathcal{Q}^\mu, \mathcal{T}^{\mu\nu}$ derivative expansion in terms of $T, u^\mu, \mu, \partial_\mu$, for example:

$$\mathcal{E} = \varepsilon + \varepsilon_1 \dot{T}/T + \varepsilon_2 \partial_\mu u^\mu + \varepsilon_3 u^\mu \partial_\mu(\mu/T) + O(\partial^2). \tag{15}$$

with $\dot{T} = u^\mu \partial_\mu T$ etc.

Analogy to quantum field theory.

This approach has successfully been applied to derive causal, stable, and well-posed first order hydrodynamics:

F.S. Bemfica, M.M. Disconzi, J. Noronha, Phys. Rev. D98 (2018) 104064
F.S. Bemfica, M.M. Disconzi, J. Noronha, Phys. Rev. D100 (2019) 104020
P. Kovtun, JHEP 1910 (2019) 034

derivation from the kinetic theory, momentum dependent RTA

R. Biswas, S. Mitra, V. Roy, Phys. Rev. D 106 (2022) L011501, Phys. Lett. B838 (2023) 137725

4.2 BDNK vs. IS

With baryon chemical potential neglected,
BDNK yields 4 equations of the second order,
while IS gives 10 equations of the first order.
Clearly, the two approaches cannot be in general equivalent. However, the equivalence may be found in the cases obeying special symmetry constraints, for example, for boost-invariant systems with constant relaxation time.

A. Das, WF, J. Noronha, R. Ryblewski, Phys. Lett. B806 (2020) 135525
A. Das, WF, R. Ryblewski, Phys. Rev. D102 (2020) 031501
A. Das, WF, R. Ryblewski, Phys. Rev. D103 (2021) 014011

for a boost-invariant expansion of a fluid with a conformal equation of state and constant relaxation time τ_R
$\varepsilon_1 = 4a\tau_R \quad \varepsilon_2 = \frac{4}{3}a\tau_R$
a is a constant in the equation of state

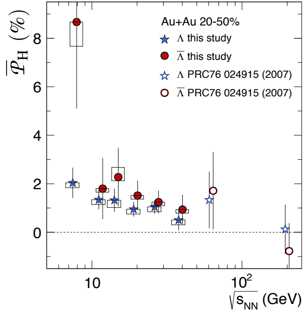
5 Hydrodynamics with spin

5.1 Is QGP the most vortical fluid?

First positive measurements of Λ spin polarization

Non-central heavy-ion collisions create fireballs with large global angular momenta which may generate a spin polarization of the hot and dense matter in a way similar to the Einstein-de Haas and Barnett effects
Much effort has recently been invested in studies of polarization and spin dynamics of particles produced in high-energy nuclear collisions, both from the experimental and theoretical point of view
L. Adamczyk et al. (**STAR**), (2017), **Nature 548 (2017) 62-65**, arXiv:1701.06657 (nucl-ex)

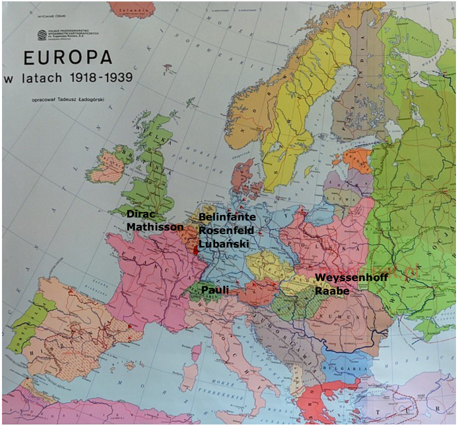
Global Λ hyperon polarization in nuclear collisions: evidence for the most vortical fluid
www.sciencenews.org/article/smashing-gold-ions-creates-most-swirly-fluid-ever



5.2 Weyssenhoff's spinning fluid

The Weyssenhoff circle, years 1930s - 1940s
J. Weyssenhoff and A. Raabe, Acta Phys. Pol. 9 (1947) 7

Jan Weyssenhoff
1889-1972



$j^\mu = n u^\mu$

1) conservation of energy and momentum with an asymmetric energy-momentum tensor

$$\boxed{T^{\mu\nu}(x) = g^\mu(x) u^\nu(x)}, \quad \partial_\nu T^{\mu\nu}(x) = 0$$

u^μ is the four-velocity of the fluid element, while g^μ is the density of four-momentum with the notation $\partial_\nu (f u^\nu) \equiv Df$ we may write $Dg^\mu = 0$

2) conservation of total angular momentum $J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu}$ (orbital and spin parts)

$$\boxed{L^{\lambda,\mu\nu}(x) = x^\mu T^{\nu\lambda}(x) - x^\nu T^{\mu\lambda}(x), \quad S^{\lambda,\mu\nu}(x) = s^{\mu\nu}(x) u^\lambda(x)}$$

$s^{\mu\nu} = -s^{\nu\mu}$ describes the spin density

$$\partial_\lambda J^{\lambda,\mu\nu} = 0 \to Ds^{\mu\nu} = g^\mu u^\nu - g^\nu u^\mu$$

3) 10 equations for 13 unknown functions: g^μ, $s^{\mu\nu}$ and u^i ($i = 1, 2, 3$)
additional constraint has been adopted, the Frenkel (or Weyssenhoff) condition
$s^{\mu\nu} u_\mu = 0$

popular use now in modified gravity theories with torsion, Cartan-Einstein, ...

5.3 Spin hydrodynamics

revival of the ideas of spinning fluids

WF, B. Friman, A. Jaiswal, E. Speranza, Phys. Rev. C97 (2018) 041901
idea of a spin chemical potential $\omega_{\mu\nu}$ introduced, as an independent hydrodynamic variable

general concept of hydrodynamics with spin: conservation of energy, linear momentum, total angular momentum, and charge:

$$\partial_\mu T^{\mu\nu}[\beta^\alpha, \omega^{\alpha\beta}, \xi] = 0, \tag{16}$$

$$\partial_\lambda J^{\lambda,\mu\nu}[\beta^\alpha, \omega^{\alpha\beta}, \xi] = 0, \tag{17}$$

$$\partial_\mu j^\mu[\beta^\alpha, \omega^{\alpha\beta}, \xi] = 0. \tag{18}$$

Here $\beta^\alpha = u^\alpha/T$, $\xi = \mu/T$, where μ is the chemical potential

$\omega^{\alpha\beta}$ - new chemical potential connected with the angular momentum conservation
$J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu}$ - sum of the orbital and spin parts

$$\partial_\lambda J^{\lambda,\mu\nu} = 0 \iff \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu} \tag{19}$$

spin-orbit interaction, quantum energy-momentum tensors have asymetric parts
conservation of angular momentum for particle with spin is non-trivial

Pseudo-gauge freedom

taken from F. Becattini

Pseudo-gauge transformations with a superpotential $\widehat{\Phi}$

F. W. Hehl, Rept. Math. Phys. 9 (1976) 55 (see also F. B., L. Tinti Phys. Rev. D 84 (2011) 025013)

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha \left(\widehat{\Phi}^{\alpha,\mu\nu} - \widehat{\Phi}^{\mu,\alpha\nu} - \widehat{\Phi}^{\nu,\alpha\mu} \right)$$
$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = \widehat{\mathcal{S}}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu}$$

They leave the conservation equations and spacial integrals (=generators, or total energy, momentum and angular momentum operators) invariant

EXAMPLE: Belinfante symmetrization

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha \left(\widehat{\mathcal{S}}^{\alpha,\mu\nu} - \widehat{\mathcal{S}}^{\mu,\alpha\nu} - \widehat{\mathcal{S}}^{\nu,\alpha\mu} \right)$$
$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = 0$$

Free Dirac field:

$$\widehat{T}^{\mu\nu} = \frac{1}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi$$
$$\widehat{\mathcal{S}}^{\lambda,\mu\nu} = \frac{1}{2}\overline{\Psi}\{\gamma^\lambda, \Sigma^{\mu\nu}\}\Psi = \frac{i}{8}\overline{\Psi}\{\gamma^\lambda[\gamma^\mu,\gamma^\nu]\}\Psi$$

Canonical pseudo-gauge

$$\widehat{T}'^{\mu\nu} = \frac{i}{4}\left[\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi + \overline{\Psi}\gamma^\nu \overleftrightarrow{\partial}^\mu \Psi\right]$$
$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = 0$$

Belinfante pseudo-gauge

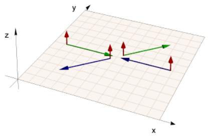
Many different formulations:

- **F. Becattini et al.**
  - spin degrees of freedom are equilibrated, spin-orbit coupling interaction included, asymmetric energy-momentum tensor, the spin chemical potential is equal to thermal vorticity
    $\omega_{\mu\nu} = -1/2\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right)$
    the spin chemical potential is not independent
  - standard (dissipative) hydro is used, $\omega_{\mu\nu}$ determined by the standard hydrodynamic variables such as T and u^μ
  - extension to include the effects of the shear stress tensor
  - forms of global and local distribution functions obtained from QFT (Dirac field under rotation and acceleration)
- **S. Bhadury, WF, A. Jaiswal, A. Kumar, R. Ryblewski, R. Singh (Kraków-NISER framework)**
  - spin tensor is separately conserved, makes sense for s-wave dominated scattering, one can use the equation
    $\partial_\lambda S^{\lambda,\mu\nu} = 0$
    $\omega_{\mu\nu}$ plays a role of the traditional Lagrange multiplier
  - the forms of the energy-momentum and spin tensor are those derived by de Groot, van Leuveen, van Weert (canonical book on the relativistic kinetic theory)
  - energy-momentum evolution has no correction from spin, the spin dynamics can be considered in a given hydrodynamic background
  - inclusion of dissipation by using an RTA collisional integral S. Bhadury, yesterday's talk
  - inclusion of magnetic fields. A. Jaiswal, yesterday's talk

Review: Relativistic hydrodynamics for spin-polarized fluids

WF, A. Kumar, R. Ryblewski, Prog.Part.Nucl.Phys. 108 (2019) 103709

- N. Weickgenannt, E. Speranza, X.-l. Sheng, Q. Wang, D. Wagner and D. H. Rischke
  - classical treatment of spin with non-local effects



local collision: with $b = 0$, s-wave scattering (as in figure), non-local collisions $b \neq 0, l \neq 0$ (higher partial waves)
standard Boltzmann equation uses coarse graining, which should be "undone".

  - the method of moments (a lá Denicol, Molnar, Niemi, Rischke - DNMR method) used to derive hydrodynamic equations
  - extensions to spin-1 particles

- K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, H. Taya
  - general structures of the energy-momentum and spin tensor considered, which are allowed by the symmetries, symmetric and asymmetric parts of the energy-momentum tensor included

    $$T^{\mu\nu}_{1s} = h^\mu u^\nu - h^\nu u^\mu + \pi^{\mu\nu}$$

    $$T^{\mu\nu}_{1a} = q^\mu u^\nu - q^\nu u^\mu + \phi^{\mu\nu}$$
  - the form of the dissipative terms determined by the entropy-production analysis - in a very similar way to the Israel-Stewart method (restricted to the first-order theory)
  - connection to the original Weyssenhoff's idea + thermodynamic identities

    $S^{\lambda,\mu\nu} = u^\lambda S^{\mu\nu}$ dubbed nowadays a phenomenological form (pseudo-gauge)
    essentially different from the canonical form that is totally antisymmetric in Lorentz indices

    $$\varepsilon + P = Ts + \omega_{\alpha\beta} S^{\alpha\beta}, \quad dP = sdT + S^{\alpha\beta} d\omega_{\alpha\beta}$$
  - pseudo-gauge equivalence?
    A. Daher, A. Das, WF, RR, arXiv:2202.12609 : no pseudo-gauge equivalence with the canonical form, difference by a derivative term only

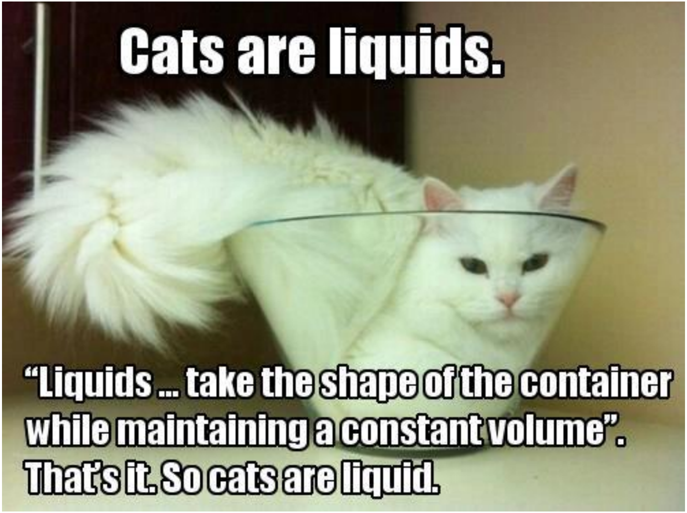
    K. Fukushima and S. Pu, PLB817 (2021) 136346: phenomenological and Belinfante formulations equivalent, if

    the same thermodynamic relations are assumed

- several other approaches, in particular for massless fermions, ...

6 Summary

- Golden era of heavy-ion collisions during the first runs of RHIC, 2000-2010, continuation at the LHC
- Enormous progress in both experiment and theory
- Great success of statistical methods (thermal models) and hydrodynamics
- Completely new POV has been established on relativistic hydrodynamics that is treated now like an effective theory of systems approaching equilibrium


Cats are liquids.
"Liquids ... take the shape of the container while maintaining a constant volume".
That's it. So cats are liquid.