Far-off-equilibrium expansion of a quark-gluon gas and the second law of thermodynamics

Chandrodoy Chattopadhyay

North Carolina State University

International Conference on the Physics and Astrophysics of Quark Gluon Plasma

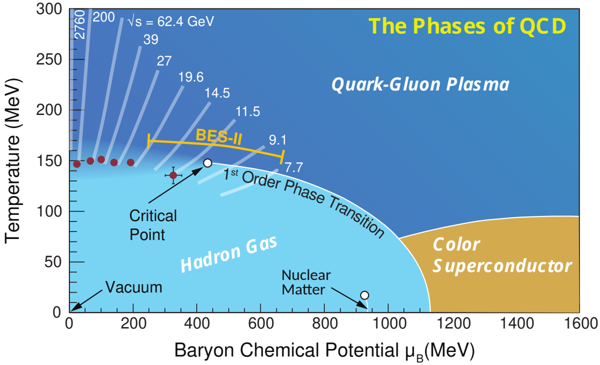
Puri, February 7, 2023

Collaborators: Ulrich Heinz (OSU) and Thomas Schäfer (NCSU)

(Based on arXiv:2209.10483)

NC STATE

Introduction



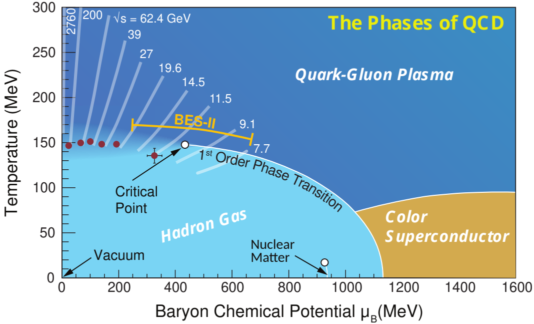
(Figure by A. Aprahamian et al.)

- Understanding the QCD phase diagram is one of the main goals of heavy-ion collisions.
- It is conjectured that a first-order phase transition line exists in QCD phase diagram which terminates at a critical point [Stephanov, Rajagopal, Shuryak '98]; under intense investigation at Beam Energy Scan (BES) programs at RHIC.
- Most dramatic effects of a critical point should be manifested in fluctuation observables [Stephanov, Rajagopal, Shuryak, Hatta, Nu Xu, and others].

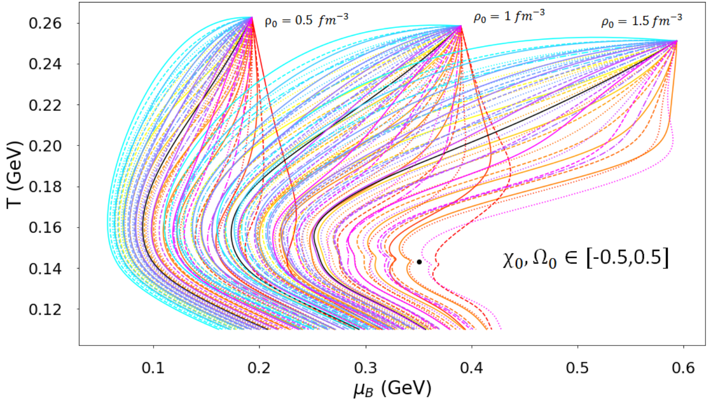
Introduction

- Several complications: Explosive dynamics of fireball, fluctuations are not equilibrated. Need models to describe intertwined dynamics of critical fluctuations and bulk evolution [Stephanov, Yin '18, Akamatsu, Teaney et al. '18, and several others]
- It is crucial to accurately model the bulk evolution in regions of finite net baryon density.

- Single collision energy does not yield a single phase trajectory.
- Different parts of the fireball have different (T, μ_B); they follow different expansion trajectories.
- Dissipative effects shuffles trajectories around.



A swath of phase trajectories (Travis Dore et al. PRD 102 (2020), 074017)



(Dore et al. PRD 102 (2020), 074017)

- Results from second-order hydro. Assumptions: Bjorken flow with Lattice QCD based EoS. ($\chi = \pi/(e+P)$, $\Omega = \Pi/(e+P)$)
- But... some hydro trajectories appear to violate the second law of thermodynamics. Need a deeper understanding. Main motivation of this work.

Setup: Hydrodynamics of a weakly interacting QG-gas

- Consider a weakly interacting gas of quarks, anti-quarks, and gluons.
- Assume a kinetic description in terms of single-particle distribution functions, $f^i(x,p)$; 'i' denotes species.
- Evolution of $f^i(x,p)$ governed by Boltzmann equation:

$$p_i^\mu \, \partial_\mu f^i = \mathcal{C}[f^i]$$

- Approximate collisional kernel of the relaxation type [Andersen & Witting '74]:

$$\mathcal{C}[f^i] \approx -\frac{u \cdot p_i}{\tau_R} \left(f^i - f^i_{eq}\right),$$

τ_R is relaxation time for local equilibration, $u^\mu(x)$ is local fluid velocity.

- f_{eq} are given by Fermi-Dirac (for quarks, anti-quarks) or Bose-Einstein (for gluons) distributions in fluid rest frame. For eg.,

$$f^q_{eq} = \left[\exp\left(\beta(u \cdot p) - \alpha\right)\right]^{-1}, \text{ where } \beta = 1/T, \alpha = \mu/T.$$

Hydro from kinetic theory

- Using $f(x,p)$ one obtains conserved currents of hydro:

$$T^{\mu\nu}(x) = \sum_i \int dP_i\, p_i^\mu\, p_i^\nu\, f^i = e\, u^\mu\, u^\nu - (P + \Pi)\, \Delta^{\mu\nu} + \pi^{\mu\nu},$$

$$N^\mu(x) = \int dP_q\, p_q^\mu\, \left(f^q - f^{\bar{q}}\right) = n\, u^\mu + n^\mu$$

- The off-equilibrium corrections to $T^{\mu\nu}$ stem from $\delta f^i \equiv f^i - f^i_{eq}$:

$$\Pi = -\frac{\Delta_{\alpha\beta}}{3} \sum_{i=1}^{3} \int dP_i\, p_i^\alpha\, p_i^\beta\, \delta f^i,$$

$$\pi^{\mu\nu} = \sum_{i=1}^{3} \int dP_i\, p_i^{\langle\mu}\, p_i^{\nu\rangle}\, \delta f^i,$$

where $A^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta}\, A^{\alpha\beta}$ with the double symmetric, transverse, and orthogonal (to u^μ) projector,

$$\Delta^{\mu\nu}_{\alpha\beta} = \left(\Delta^\mu_\alpha\, \Delta^\nu_\beta + \Delta^\mu_\beta\, \Delta^\nu_\alpha\right)/2 - \Delta^{\mu\nu}\, \Delta_{\alpha\beta}/3.$$

Hydro from kinetic theory

- To obtain hydro, one needs to know δf^i.
- Start from RTA Boltzmann equation,

$$p_i^\mu \, \partial_\mu f^i = -\frac{u \cdot p_i}{\tau_R} \, \left(f^i - f^i_{eq}\right),$$

and assume small Knudsen number, i.e., $\tau_R \, \frac{\partial}{\partial x^\mu}$ is of $\mathcal{O}(\epsilon)$.

- Consider a perturbative series,

$$f^i = f^i_0 + \epsilon \, \delta f^i_1 + \epsilon^2 \, \delta f^i_2 + \cdots$$

and solve order by order in ϵ:

$$f^i_0 = f^i_{eq}, \quad \delta f^i_1 = -\frac{\tau_R}{(u \cdot p_i)} \, p_i^\mu \, \partial_\mu f^i_{eq},$$

$$\delta f^i_2 = \frac{\tau_R}{(u \cdot p_i)} \, p_i^\mu \, p_i^\nu \, \partial_\mu \left(\frac{\tau_R}{u \cdot p_i} \partial_\nu f^i_{eq}\right).$$

Hydro from kinetic theory

- Calculating δf^i up to second-order in velocity gradients we obtain evolution equations of bulk and shear stress tensors:

$$\dot{\Pi} = -\frac{\Pi}{\tau_R} - \beta_\Pi\,\theta - \delta_{\Pi\Pi}\,\Pi\,\theta + \lambda_{\Pi\pi}\,\pi^{\mu\nu}\,\sigma_{\mu\nu},$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = -\frac{\pi^{\mu\nu}}{\tau_R} + 2\,\beta_\pi\,\sigma^{\mu\nu} + 2\,\pi_\gamma^{\langle\mu}\,\omega^{\nu\rangle\gamma} - \tau_{\pi\pi}\,\pi_\gamma^{\langle\mu}\,\sigma^{\nu\rangle\gamma} - \delta_{\pi\pi}\,\pi^{\mu\nu}\,\theta + \lambda_{\pi\Pi}\,\Pi\,\sigma^{\mu\nu},$$

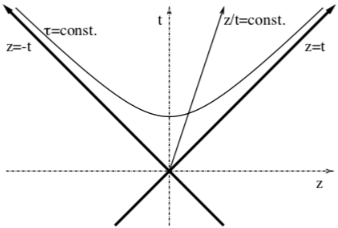
where the coefficients $(\beta_\Pi, \delta_{\Pi\Pi}, \lambda_{\Pi\pi}, \cdots)$ are functions of (T, μ); obtained in arXiv:2209.10483 [C.C., Heinz, Schäfer].

- Standard definitions: $\dot{A} = u^\mu \partial_\mu A$ (time-derivative), $\theta = \partial_\mu u^\mu$ (expansion rate), $\nabla^\mu = \Delta^{\mu\nu}\,\partial_\nu$ (space-like derivative), velocity stress tensor $\sigma^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\nabla^\alpha u^\beta$, vorticity $\omega^{\mu\nu} = (\nabla^\mu u^\nu - \nabla^\nu u^\mu)$.

Application: Bjorken flow [J.D. Bjorken, PRD, 27, 140 (1983)]

- Bjorken flow is valid during the early stages of ultra-relativistic heavy-ion collisions.
- The fluid is assumed to be homogeneous in $(x - y)$ direction.
- The medium expands boost-invariantly along beam $(z-)$ direction: $v^x = 0, \; v^y = 0, \; v^z = z/t$.

- Switch to Milne coordinates $(\tau, x_\perp, \phi, \eta_s)$ where $\tau \equiv \sqrt{t^2 - z^2}$, and $\eta_s \equiv \tanh^{-1}(z/t)$.
- Fluid appears static, $u^\mu = (1, 0, 0, 0)$. However, has finite expansion rate, $\theta = 1/\tau$.



Consequences of Bjorken symmetries

- Shear stress tensor has only one independent quantity: $\pi^{\mu\nu} = \mathrm{diag}(0, \pi/2, \pi/2, -\pi/\tau^2)$. All functions depend solely on proper time τ. Also, vorticity $\omega^{\mu\nu} = 0$.
- The evolution equations for energy density, number density and shear stress component π are:

$$\frac{de}{d\tau} = -\frac{1}{\tau}\left(e + P + \Pi - \pi\right), \quad (1)$$

$$\frac{dn}{d\tau} = -\frac{n}{\tau}, \quad (2)$$

$$\frac{d\Pi}{d\tau} = -\frac{\Pi}{\tau_R} - \frac{\beta_\Pi}{\tau} - \delta_{\Pi\Pi}\frac{\Pi}{\tau} + \lambda_{\Pi\pi}\frac{\pi}{\tau}, \quad (3)$$

$$\frac{d\pi}{d\tau} = -\frac{\pi}{\tau_R} + \frac{4}{3}\frac{\beta_\pi}{\tau} - \left(\frac{1}{3}\tau_{\pi\pi} + \delta_{\pi\pi}\right)\frac{\pi}{\tau} + \frac{2}{3}\lambda_{\pi\Pi}\frac{\Pi}{\tau}, \quad (4)$$

Case I: Conformal Bjorken flow

- Consider massless quarks and anti-quarks, bulk viscous pressure $\Pi = 0$, relaxation time $\tau_R \propto 1/T$.
- The evolution equations are,

$$\frac{de}{d\tau} = -\frac{1}{\tau}(e + P - \pi), \quad \frac{dn}{d\tau} = -\frac{n}{\tau},$$
$$\frac{d\pi}{d\tau} = -\frac{\pi}{\tau_R} + \frac{4}{3}\frac{\beta_\pi}{\tau} - \left(\frac{1}{3}\tau_{\pi\pi} + \delta_{\pi\pi}\right)\frac{\pi}{\tau},$$

where the coefficients simplify: $\beta_\pi = 4P/5$, $\tau_{\pi\pi} = 10/7$, $\delta_{\pi\pi} = 4/3$.

- To relate $(e, n) \to (T, \mu)$ we use the conformal relations,

$$e(T,\mu) = T^4\left[\frac{(4g_g + 7g_q)\pi^2}{120} + \frac{g_q}{4}\left(\frac{\mu}{T}\right)^2 + \frac{g_q}{8\pi^2}\left(\frac{\mu}{T}\right)^4\right] = 3P(T,\mu),$$
$$n(T,\mu) = T^3\left[\frac{g_q}{6}\left(\frac{\mu}{T}\right) + \frac{g_q}{6\pi^2}\left(\frac{\mu}{T}\right)^3\right].$$

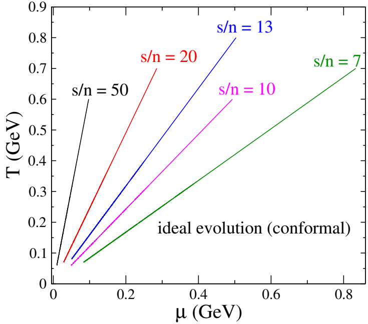
Case Ia: Ideal conformal Bjorken flow

- We now set shear to zero:

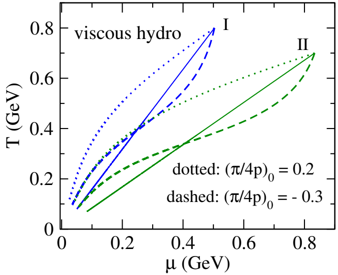
$$\frac{de}{d\tau} = -\frac{1}{\tau}(e+P), \quad \frac{dn}{d\tau} = -\frac{n}{\tau},$$
$$\Longrightarrow \frac{ds}{d\tau} = -\frac{s}{\tau},$$

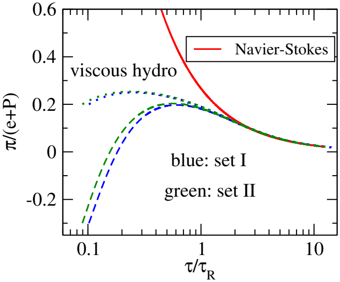
where entropy density $s = (e + P - \mu n)/T$.

- Simple scaling solutions: $s(\tau) \propto 1/\tau$, $n(\tau) \propto 1/\tau$ such that entropy per baryon s/n is fixed.
- Due to conformality, fixed $s/n \Longrightarrow$ fixed μ/T.
- s/n increases from right to left in the phase diagram.



- Include shear in the evolution dynamics.
- Dotted lines initialised with positive $\pi/(4P)$; they lie to left of ideal trajectory. Expected: dissipation leads to entropy generation.
- But dashed lines with $(\pi/4P)_0 < 0$ move to the right for some time. Violation of the second law in hydrodynamics?
- Perhaps such trajectories do not exist in the microscopic (kinetic) theory.



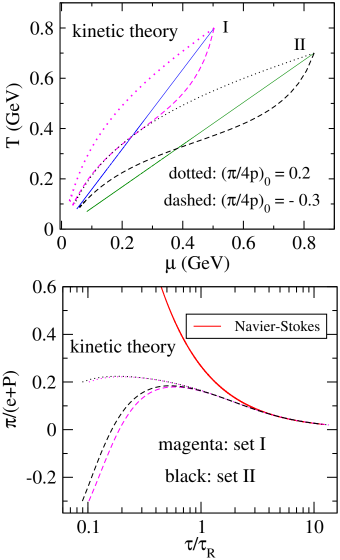


Viscous dynamics: Conformal kinetic theory

- Solve the Boltzmann equation:

$$\frac{\partial f^i}{\partial \tau} = -\frac{f^i - f^i_{eq}}{\tau_R}$$

- Dotted lines initialised with positive $\pi/(4P)$; they lie to left of ideal trajectory.
- But dashed lines with $(\pi/4P)_0 < 0$ lie to the right for some time.
- Such trajectories are also present in the microscopic theory!



Toward a resolution

- Statement of the second law: $\partial_\mu S^\mu \geq 0$.
- Thus far we have assumed $S^\mu = s_{eq}\, u^\mu$ with $s_{eq} = (e + P - \mu n)/T$.
- But is it justified when the system deviates substantially from local equilibrium?
- Need an expression for non-equilibrium entropy.

Non-equilibrium entropy current

- Start from Boltzmann's H-function,

$$S^\mu = -\sum_i g_i \int dP\, p^\mu\, \phi_i[f^i],$$

where $i = 1, 2, 3$ labels quarks, anti-quarks, and gluons. The functions $\phi_i[f^i]$ are given as,

$$\phi_i[f^i] = f^i \ln(f^i) - \frac{1+\theta_i f^i}{\theta_i} \ln(1+\theta_i f^i),$$

with $\theta_1 = \theta_2 = -1$ (Fermi-Dirac) and $\theta_3 = 1$ (Bose-Einstein)

- Writing $f^i = f^i_{eq} + \delta f^i$, we expand entropy current to second-order in δf^i:

$$S^\mu = s_{eq}\, u^\mu - \alpha n^\mu - \sum_i g_i \int dP\, p^\mu\, \frac{\phi_i^{''}[f^i]}{2}\, (\delta f^i)^2.$$

Second-order conformal entropy current

- δf^i obtained by perturbatively solving the RTA Boltzmann equation; for example,

$$\frac{\delta f^q}{\mathcal{F}^q_{eq}} = \mathcal{A}^{\mu\nu}_p \, \pi_{\mu\nu} + \mathcal{B} \, p^\mu n_\mu, \quad \text{where } \mathcal{F}^q_{eq} = f^q_{eq} \, \left(1 - f^q_{eq}\right),$$

[Jaiswal, Friman, Redlich, '15]

- The second-order entropy current is found to be,

$$S^\mu = s_{\rm eq} \, u^\mu - \alpha \, n^\mu - \frac{\beta}{4\beta_\pi} \, u^\mu \, \pi^{\alpha\beta} \, \pi_{\alpha\beta} + c_{nn} \, u^\mu \, n^\alpha \, n_\alpha + c_{n\pi} \, \pi^{\mu\alpha} \, n_\alpha;$$

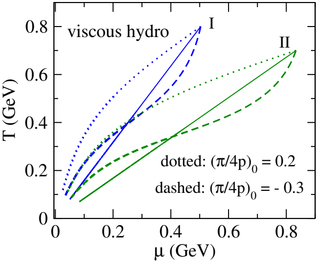
the coefficients $(\beta_\pi, c_{nn}, c_{n\pi})$ are derived in arXiv:2209.10483 [C.C., Heinz, Schäfer].

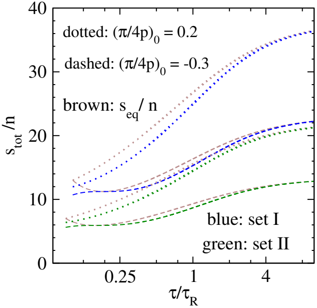
- Note, entropy flux in fluid rest frame, $S^{\langle\mu\rangle} = \Delta^\mu_\nu S^\nu$ is not necessarily along baryon diffusion. In Bjorken flow,

$$s = s_{eq} - \frac{3\beta}{8\beta_\pi} \, \pi^2.$$

Second law in conformal hydrodynamics

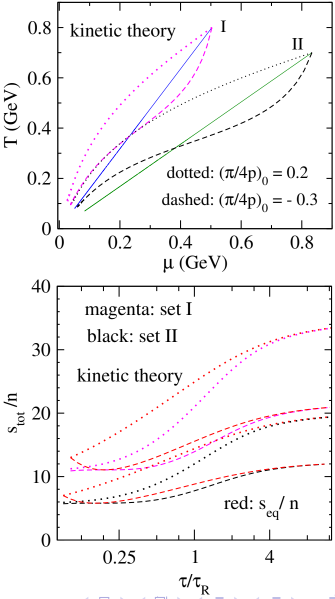
- Brown curves (lower panel) denote evolution of s_{eq}/n.
- For dotted brown curves (positive initial shear), s_{eq}/n always increase.
- For the dashed brown curves, (s_{eq}/n) decreases initially. However, the dashed curves start from a lower total s/n.
- The total entropy per baryon of the dashed curves never decrease.
- All the hydro trajectories are consistent with the second law!



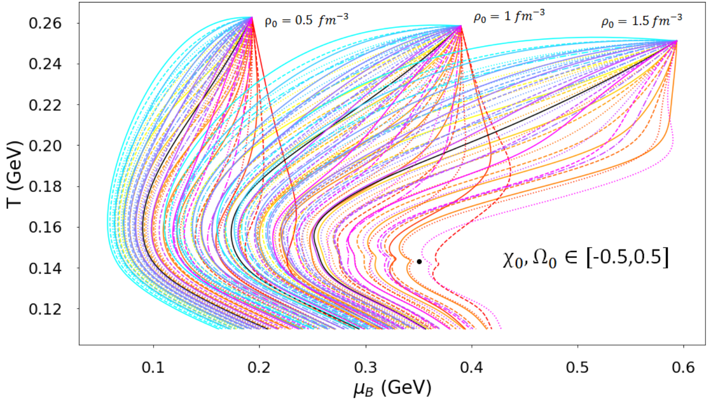


Second law in conformal kinetic theory

- Red curves (lower panel) denote evolution of s_{eq}/n.
- For dotted red curves (positive initial shear), the s_{eq}/n always increase.
- For the dashed red curves, s_{eq}/n decreases initially.
- Key conclusion: Second law demands (i) $s/n \leq s_{eq}/n$ and (ii) $d(s/n)/d\tau \geq 0$; but s_{eq}/n need not increase.



Coming back to the phase trajectories of Dore et al.



(T. Dore et al. PRD 102 (2020), 074017)

- Assumption: Second-order hydro for Bjorken flow with Lattice QCD based EoS. ($\chi = \pi/(e+P)$, $\Omega = \Pi/(e+P)$)
- Is it also possible that the chemical potential of an expanding quark-gluon gas increases instead of decreasing?

Case II: Non-conformal dynamics

- We break conformal symmetry by making the quarks and anti-quarks massive $\Longrightarrow$ non-vanishing bulk viscous pressure.

Case IIa: Ideal non-conformal Bjorken flow

- Ideal hydro for Bjorken flow:

$$\frac{de}{d\tau} = -\frac{1}{\tau}(e + P),$$
$$\frac{dn}{d\tau} = -\frac{n}{\tau}.$$

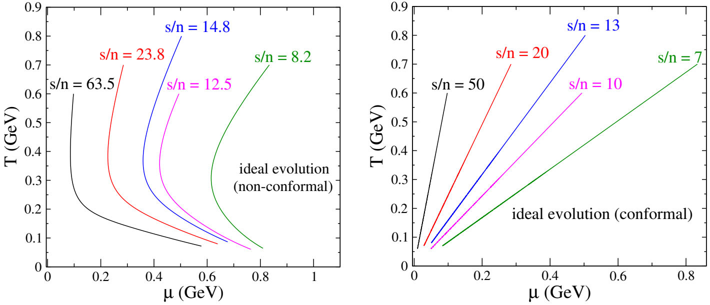
- The conversion from $(e, n) \to (T, \mu)$ is implemented by Landau matching:

$$e = \sum_i \int \frac{d^3 p_i}{(2\pi)^3} E_p^i f_{eq}^i(T, \mu),$$
$$n = \int \frac{d^3 p}{(2\pi)^3} \left(f_{eq}^q(T, \mu) - f_{eq}^{\bar{q}}(T, \mu) \right).$$

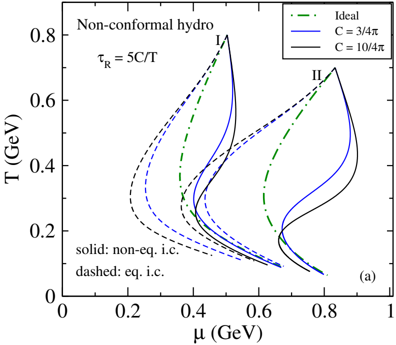
- For purposes of demonstration we chose a large quark and anti-quark mass, $m_q = m_{\bar{q}} = 1$ GeV.

Case IIa: Ideal conformal vs non-conformal

- Unlike conformal case, constant s/n does not imply constant μ/T. Non-conformality leads to $s/n = f(\mu/T, m/T)$.

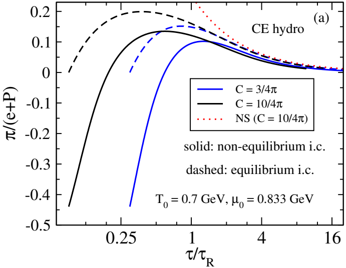


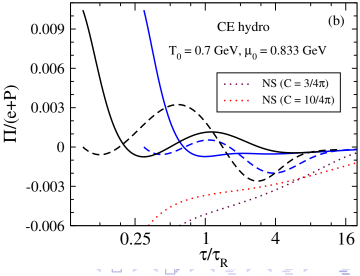
- At high T, EoS dominated by quarks, anti-quarks, and gluons.
- At low T, EoS dominated by quarks.
- As $T \to 0$, Fermi statistics of quarks imply $\mu \to m$.



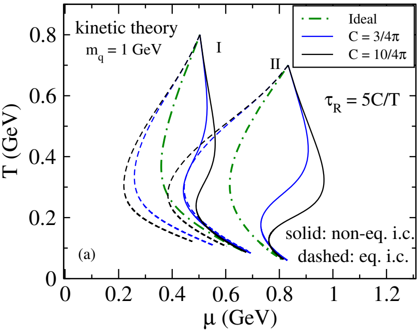
- Solid trajectories lie to right of ideal curves.
- They have increasing μ at early times!
- Problem or feature?

- Evolution of shear and bulk stresses:



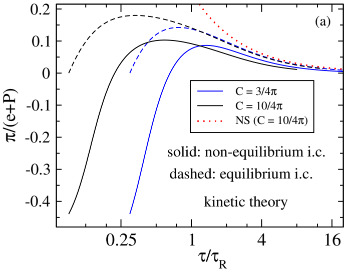


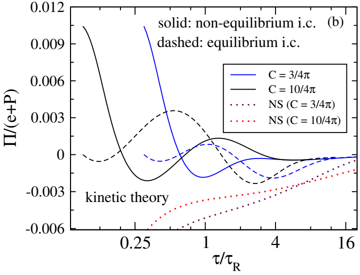
Non-conformal kinetic theory arXiv:2209.10483 [C.C., Heinz, Schäfer]



- Trajectories with increasing μ also present in kinetic theory!
- Not a problem, a feature.

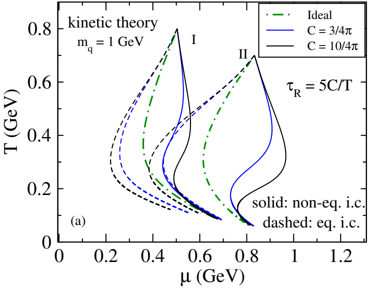
- Evolution of shear and bulk stresses:

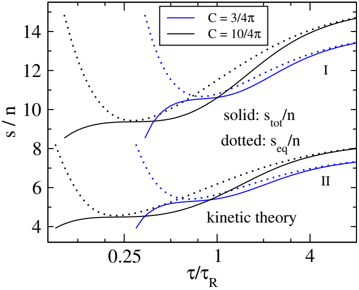




Second law in non-conformal kinetic theory [C.C., Heinz, Schäfer]

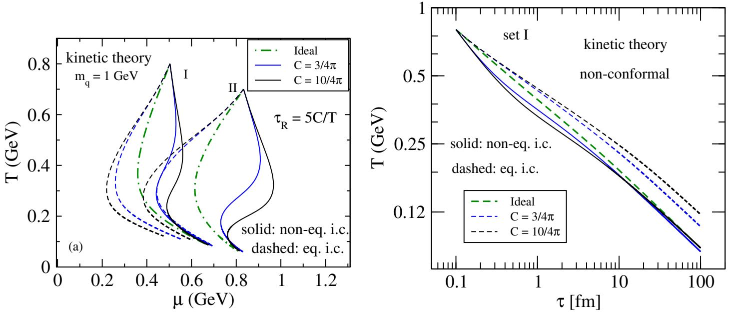
- Dotted curves (lower panel) denote evolution of s_{eq}/n.
- Although s_{eq}/n decreases, the total s/n computed using Boltzmann's H-function does not.
- 3 distinct regimes of s/n evolution:
  - early increase of s/n : expansion driven isotropization,
  - intermediate plateau where $s/n \approx s_{eq}/n$ (free-streaming),
  - eventual merging with s_{eq}/n (interaction driven).





Viscous cooling! arXiv:2209.10483 [C.C., Heinz, Schäfer]

- Usually dissipative fluxes causes viscous heating $\implies$ Temperature falls slower than in ideal evolution.



- However, in dissipative dynamics, temperature may decrease faster than even in the ideal case.
- In Bjorken, this happens when the effective longitudinal pressure $P_L > P \implies \pi - \Pi < 0$. A manifestation of decreasing s_{eq}/n:

$$\frac{d(s_{eq}/n)}{d\tau} = \frac{\pi - \Pi}{\tau_0\, n_0\, T}.$$

Summary and Conclusions

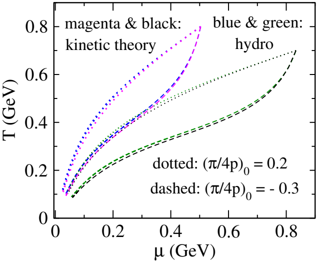
- Derived second-order hydrodynamics of a weakly interacting non-conformal gas of quarks and gluons using kinetic theory.
- Dissipative fluxes were found to shuffle around phase trajectories; they exhibit substantial sensitivity to initial shear and bulk viscous stresses.
- Some of the hydrodynamic trajectories appear to the violate the second law at first sight. By deriving a second-order conformal entropy current, it was shown that such trajectories were in complete agreement with the second law.
- An in-depth analysis of non-equilibrium entropy production in kinetic theory was presented.
- A novel effect (viscous cooling) was pointed out where a dissipative system cools faster than in the inviscid case.
- It would be important to explore these features in flow profiles with less restricted symmetries.

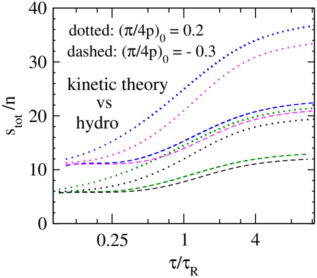
Backup Slide 1: Applicability of classical kinetic theory [Jeon and Heinz, arXiv:1503.03931 (2015)]

- Hydro formulated as a series in velocity gradients: $\pi^{ij} \sim \eta\partial^i v^j$, $\Pi \sim -\zeta\partial \cdot v$.
- Three scales: Two microscopic: $l_{mfp} \sim 1/(\sigma vn)$, thermal wavelength $l_{th} \sim 1/T$, one macroscopic $1/L \sim \partial \cdot u$.
- $l_{mfp}/l_{th} \sim \eta/s$, ζ/s, $T\kappa/s$
- Hydro applicable whenever microscopic and macroscopic scales are well-separated: $l_{mfp}\,\partial \cdot u \equiv Kn < 1$
  - Dilute gas regime: $l_{mfp}/l_{th} \sim \eta/s \gg 1$; Weakly coupled regime, Boltzmann equation applicable (on-shell particles).
  - Dense gas regime: $\eta/s \sim 1$; quasi-particle description in terms of Wigner functions.
  - Liquid regime: $\eta/s \ll 1$; strong-coupling regime, no valid kinetic description.

Backup slide 2: Conformal hydro vs kinetic theory

- Hydrodynamic (second-order) entropy per baryon vs exact entropy per baryon obtained using kinetic theory:





- Hydro over-estimates the entropy per baryon produced by ~ 10 percent.

Backup 3: Non-equilibrium entropy

- The canonical entropy $S = -\sum_i p_i \ln(p_i)$ for a continuous distribution:

$$S = -\int \frac{d^{3N}x\, d^{3N}p}{N!}\, \rho \ln(\rho),$$

where,

$$\rho(x_1, \cdots, x_N, p_1, \cdots, p_N) = \frac{\exp(-\beta H_N(x_1, \cdots, x_N, p_1, \cdots, p_N))}{Z(T, V, N)}$$

- Due to weak interaction,

$$H_N = \sum_i H_i, \quad Z(T, V, N) = Z(T, V, 1)^N = V^N n^N / N!,$$

where n is number density. Thus,

$$S = -\frac{\beta V^N}{Z(T, V, 1)^N} \int d^{3N}p\, H(p) \exp(-\beta H(p)) - \ln(Z(T, V, N))$$

Backup 4: Non-equilibrium entropy

- For large N, $\ln(Z(T, V, N)) \approx N$. Thus,

$$S = V \int d^3p \, (\beta \, H(p) \, f_{eq} + f_{eq}) \, ,$$

  and the entropy density:

$$s = - \int d^3p \, f_{eq} \, (\ln(f_{eq}) - 1) \, .$$

- Out of equilibrium, replace $f_{eq} \to f$. Relativistic version,

$$s = - \int dP \, (u \cdot p) \, f \, (\ln(f) - 1) \, .$$