Relativistic dissipative hydrodynamics with BGK collision kernel

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Introduction

- Energy momentum tensor :

$$T^{\mu\nu} = \int \mathrm{dP}\, p^\mu p^\nu \left(f + \bar{f}\right) = \epsilon\, u^\mu u^\nu - P\Delta^{\mu\nu} + \pi^{\mu\nu}$$

- Net particle four current:

$$N^\mu = \int \mathrm{dP}\, p^\mu \left(f - \bar{f}\right) = n\, u^\mu + n^\mu$$

- Landau frame[1]: $u_\mu T^{\mu\nu} = \epsilon u^\nu$

[1] L. D. Landau and E. M. Lifshitz, Fluid Mechanics (Pergamon, New York, 1987)

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- Net particle four current:

$$N^\mu = \int \mathrm{dP}\, p^\mu \left(f - \bar{f}\right) = \left(n_0 + \delta n\right)\, u^\mu + n^\mu$$

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Introduction

- For equilibrium system:

$$\epsilon_0 = \int dP\,(u\cdot p)^2\left(f_0 + \bar{f}_0\right)$$

$$P_0 = -\frac{1}{3}\Delta_{\mu\nu}\int dP\,p^\mu p^\nu\left(f_0 + \bar{f}_0\right)$$

$$n_0 = \int dP\,(u\cdot p)\left(f_0 - \bar{f}_0\right)$$

- Maxwell -Juttner distribution function :

$$f_0 = \exp\left(-\beta(u\cdot p) + \alpha\right)$$

$$\beta = \frac{1}{T}, \quad \alpha = \frac{\mu}{T}$$

Introduction

- For out of equilibrium system:

$$\delta\epsilon = \int dP\,(u\cdot p)^2\left(\delta f + \delta\bar{f}\right)$$

$$\delta P = -\frac{1}{3}\Delta_{\alpha\beta}\int dP\,p^{\alpha}p^{\beta}\left(\delta f + \delta\bar{f}\right)$$

$$\delta n = \int dP\,(u\cdot p)\left(\delta f - \delta\bar{f}\right)$$

$$n^{\mu} = \Delta^{\mu}_{\alpha}\int dP\,p^{\alpha}\left(\delta f - \delta\bar{f}\right)$$

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int dP\,p^{\alpha}p^{\beta}\left(\delta f + \delta\bar{f}\right)$$

Where,

$$\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta} + \Delta^{\mu}_{\beta}\Delta^{\nu}_{\alpha} - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right)$$

Boltzmann Equation

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[2] James L Anderson and HR Witting. Physica, 74(3):466–488, 1974

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- Conservation equation:

$$\partial_\mu T^{\mu\nu} = \int dP \, p^\mu \, \mathcal{C}[f]; \quad \partial_\mu N^\mu = \int dP \, \mathcal{C}[f]$$

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$$\partial_{\mu}N^{\mu} = -\frac{1}{\tau_R}(n - n_0)$$
$$\partial_{\mu}T^{\mu\nu} = -\frac{u^{\nu}}{\tau_R}(\epsilon - \epsilon_0)$$

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- If, $n = n_0, \quad \epsilon = \epsilon_0$ $\rightarrow$ Matching conditions

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Boltzmann Equation

- Bhatnagar-Gross-Krook (BGK) approximation[3]:

$$p^{\mu}\partial_{\mu}f = -\frac{(u \cdot p)}{\tau_{\mathrm{R}}}\left(f - \frac{n}{n_0}f_0\right)$$

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- If, $\epsilon = \frac{n}{n_0}\epsilon_0$ $\rightarrow$ Matching condition

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Evaluation of δf

- Boltzmann Equation in BGK model: $p^\mu \partial_\mu f = -\frac{(u\cdot p)}{\tau_R}\left(f - \frac{n}{n_0} f_0\right)$
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- For first order

$$\delta f^{(1)} = \frac{\delta \epsilon}{\epsilon_0} f_0 - \frac{\tau_R}{(u \cdot p)} p^\mu \partial_\mu f_0$$

$$p^\mu \partial_\mu f_0 = \left(A_\Pi \theta + A_n p^\mu \nabla_\mu \alpha + A_\pi p^\mu p^\nu \sigma_{\mu\nu}\right) f_0$$

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$$\delta f^{(1)} = \frac{\delta\epsilon}{\epsilon_0} f_0 - \frac{\tau_R}{(u\cdot p)} p^\mu \partial_\mu f_0$$

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$$\delta f^{(1)} = \tau_R \left(B_\Pi \theta + B_n p^\mu \nabla_\mu \alpha + B_\pi p^\mu p^\nu \sigma_{\mu\nu}\right) f_0$$

$$\delta\epsilon = \tau_R \int \mathrm{dP}\, (u\cdot p)^2 \left(B_\Pi f_0 + \bar{B}_\Pi \bar{f}_0\right) \theta$$

Evaluation of δf

$$\delta f^{(1)} = \tau_{\mathrm{R}} f_0 \Bigg[\left\{ \frac{m^2 \beta}{3\,(u \cdot p)} + b_0 + (u \cdot p)\left(\chi_b - \frac{\beta}{3} \right) \right\} \theta - \left\{ \frac{1}{(u \cdot p)} - \frac{n_0}{(\epsilon_0 + P_0)} \right\} p^{\mu} \left(\nabla_{\mu} \alpha \right) + \frac{\beta p^{\mu} p^{\mu} \sigma_{\mu\nu}}{(u \cdot p)} \Bigg]$$

$$\delta \bar{f}^{(1)} = \tau_{\mathrm{R}} \bar{f}_0 \Bigg[\left\{ \frac{m^2 \beta}{3\,(u \cdot p)} + b_0 + 2\,\chi_a + (u \cdot p)\left(\chi_b - \frac{\beta}{3} \right) \right\} \theta + \left\{ \frac{1}{(u \cdot p)} + \frac{n_0}{(\epsilon_0 + P_0)} \right\} p^{\mu} \left(\nabla_{\mu} \alpha \right) + \frac{\beta p^{\mu} p^{\mu} \sigma_{\mu\nu}}{(u \cdot p)} \Bigg]$$

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$$\chi_a = \frac{I_{20}^{-}(\epsilon_0 + P_0) - I_{30}^{+} n_0}{I_{30}^{+} I_{10}^{+} - I_{20}^{-} I_{20}^{-}}, \quad \chi_b = \frac{I_{10}^{+}(\epsilon_0 + P_0) - I_{20}^{-} n_0}{I_{30}^{+} I_{10}^{+} - I_{20}^{-} I_{20}^{-}}$$

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$$\delta f^{(1)} = \tau_{\rm R} f_0 \Bigg[\left\{ \frac{m^2\beta}{3\,(u\cdot p)} + b_0 + (u\cdot p)\left(\chi_b - \frac{\beta}{3}\right) \right\}\theta - \left\{ \frac{1}{(u\cdot p)} - \frac{n_0}{(\epsilon_0 + P_0)} \right\} p^\mu\,(\nabla_\mu \alpha) + \frac{\beta p^\mu p^\mu \sigma_{\mu\nu}}{(u\cdot p)} \Bigg]$$

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$$I^{\pm}_{nq} = \frac{(-1)^q}{(2q+1)!!} \int {\rm dP}\,(u\cdot p)^{n-2q} \left(\Delta_{\alpha\beta} p^\alpha p^\beta\right)^q (f_0 \pm \bar{f}_0)$$

Transport coefficients

- For MBGK

$$\partial_\mu S^\mu = -\beta\,\Pi\,\theta - n^\mu \nabla_\mu \alpha + \beta \pi^{\mu\nu} \sigma_{\mu\nu},$$

where[4],

$$\Pi = \delta P - \frac{\chi_b}{\beta}\delta\epsilon + \frac{\chi_a}{\beta}\delta n, \quad n^\mu = \kappa \nabla^\mu \alpha, \quad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$$

[4] Pavel Kovtun. 10.1007/JHEP10(2019)034

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$$\delta n = \tau_{\rm R}\left(\chi_a + b_0\right) n_0\theta, \quad \delta\epsilon = \tau_{\rm R}\left(\chi_a + b_0\right)\epsilon_0\theta,$$

$$\delta P = \tau_{\rm R}\left[(\chi_a + b_0)P_0 + \chi_b\frac{(\epsilon_0 + P)}{\beta} - \frac{5}{3}\beta I_{32}^{+} - \frac{\chi_a n_0}{\beta}\right]\theta$$

$$\kappa = \tau_{\rm R}\left[I_{11}^{+} - \frac{n_0^2}{\beta(\epsilon_0 + P)}\right], \quad \eta = \tau_{\rm R}\,\beta\, I_{32}^{+}$$

Choice of b_0

- $\mathcal{A}_r^{\pm} = \int \mathrm{dP}\,(u\cdot p)^r \left(\delta f \pm \delta\bar{f}\right)$
- $\mathcal{A}_1^{-} = \mathcal{A}_2^{+} = 0 \implies \mathrm{RTA}$
- $\mathcal{A}_r^{+} = 0$ gives,

$$b_0^r = -\left(1/I_{r,0}^{+}\right)\left[\chi_b I_{r+1,0}^{+} - \beta I_{r+1,1}^{+} + \chi_a\left(I_{r,0}^{+} - I_{r,0}^{-}\right)\right]$$

Choice of b_0

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- 2 matching conditions for MBGK:

$$\epsilon\, n_0 = \epsilon_0\, n$$
$$\mathcal{A}_r^{+} = 0$$

Numerical Results

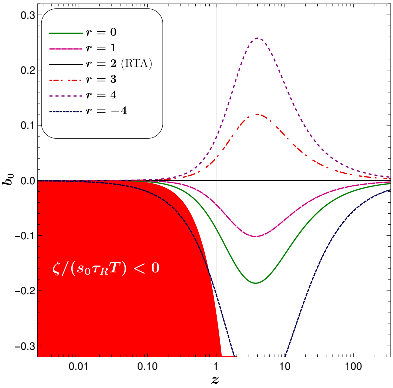

Figure 1: Dependence of the parameter b_0 on z at zero chemical potential.

- $z = \frac{m}{T}$, $\Pi = -\zeta\theta$
- s_0 = entropy density
- $b_0^{\zeta=0}|_{\alpha=0} = \frac{\chi_b(\epsilon_0+P_0)-(5/3)\beta^2 I_{32}}{\chi_b\epsilon_0-\beta P_0}$

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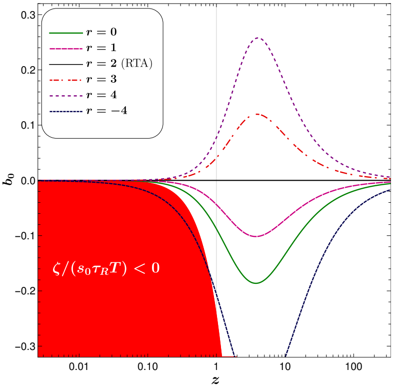

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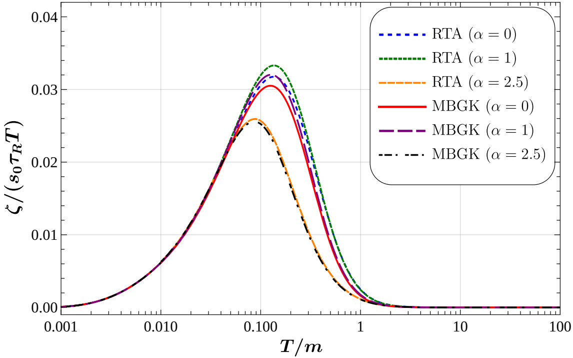

Figure 2: Dependence of $\zeta/(s_0\tau_R T)$ on T/m for various $\alpha = \mu/T$ for r=2 (RTA) and r=0(MBGK)

$$r = 2 \implies \epsilon\, n_0 = \epsilon_0\, n \quad \epsilon = \epsilon_0$$

$$\text{r=0} \implies \epsilon\, n_0 = \epsilon_0\, n \quad \epsilon - 3P = \epsilon_0 - 3P_0$$

Scaling of ζ/η

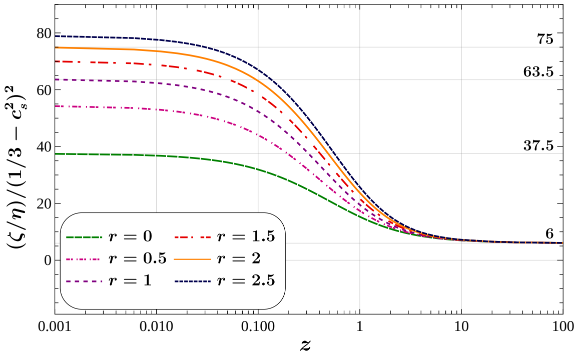

Figure 3: Variation of $(\zeta/\eta)/(1/3 - c_s^2)^2$ with respect to z for various r.

$$\lim_{z\to 0} \frac{\zeta}{\eta} = \frac{15(r^2+23r+10)}{4(r+1)} \left(\frac{1}{3} - c_s^2\right)^2 + \mathcal{O}\left(z^5\right)$$

Scaling of ζ/η

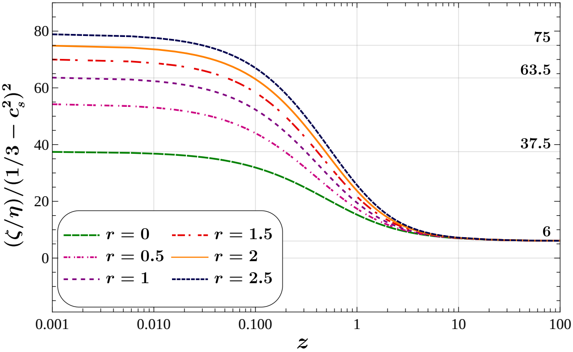

Figure 3: Variation of $(\zeta/\eta)/(1/3 - c_s^2)^2$ with respect to z for various r.

$$\lim_{z\to\infty} \frac{\zeta}{\eta} = 6\left(\frac{1}{3} - c_s^2\right)^2$$

Summary

- Formulation of relativistic dissipative hydrodynamics from BGK collision kernel is discussed.
- The theory is controlled by a free parameter related to freedom of a matching condition.
- The effect of choice of matching condition on dissipative coefficients is examined.
- Scaling properties of the ratio of coefficients of bulk viscosity to shear viscosity on the conformality measure are investigated.

Thank You For Your Attention!!

Evaluation of δf

- Boltzmann Equation in BGK model: $p^\mu \partial_\mu f = -\frac{(u\cdot p)}{\tau_R}\left(f - \frac{n}{n_0} f_0\right)$
- Matching condition $\frac{\epsilon}{\epsilon_0} = \frac{n}{n_0}$

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- For first order

$$\delta f^{(1)} = \frac{\delta\epsilon}{\epsilon_0} f_0 - \frac{\tau_R}{(u\cdot p)} p^\mu \partial_\mu f_0$$

$$p^\mu \partial_\mu f_0 = \left(A_\Pi \theta + A_n p^\mu \nabla_\mu \alpha + A_\pi p^\mu p^\nu \sigma_{\mu\nu}\right) f_0$$

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$$\delta\epsilon = \tau_{\mathrm{R}} \int \mathrm{dP}\, (u\cdot p)^2 \left(B_\Pi f_0 + \bar{B}_\Pi \bar{f}_0\right) \theta$$

Evaluation of δf

- By coefficient matching:

$$-\frac{A_\Pi}{(u\cdot p)} = B_\Pi - \frac{1}{\epsilon_0}\int \mathrm{dP}\ (u\cdot p)^2 \left(B_\Pi f_0 + \bar{B}_\Pi \bar{f}_0\right)$$

$$B_n = -\frac{A_n}{(u\cdot p)} \qquad B_\pi = -\frac{A_\pi}{(u\cdot p)}$$

Evaluation of δf

- By coefficient matching:

$$-\frac{A_\Pi}{(u\cdot p)} = B_\Pi - \frac{1}{\epsilon_0}\int \mathrm{dP}\ (u\cdot p)^2\left(B_\Pi f_0 + \bar{B}_\Pi \bar{f}_0\right)$$

$$A_\Pi = -\left[(u\cdot p)^2\left(\chi_b - \frac{\beta}{3}\right) - (u\cdot p)\,\chi_a + \frac{\beta m^2}{3}\right]$$

$$\chi_a = \frac{I_{20}^-(\epsilon_0 + P_0) - I_{30}^+ n_0}{I_{30}^+ I_{10}^+ - I_{20}^- I_{20}^-}, \quad \chi_b = \frac{I_{10}^+(\epsilon_0 + P_0) - I_{20}^- n_0}{I_{30}^+ I_{10}^+ - I_{20}^- I_{20}^-}$$

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$$I_{nq}^{\pm} = \frac{(-1)^q}{(2q+1)!!}\int \mathrm{dP}\,(u\cdot p)^{n-2q}\left(\Delta_{\alpha\beta}p^\alpha p^\beta\right)^q (f_0 \pm \bar{f}_0)$$

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$$B_\Pi = \sum_{k=-1}^{1} b_k\,(u\cdot p)^k \qquad \bar{B}_\Pi = \sum_{k=-1}^{1} \bar{b}_k\,(u\cdot p)^k$$

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$$B_\Pi = \sum_{k=-1}^{1} b_k\,(u\cdot p)^k \qquad \bar{B}_\Pi = \sum_{k=-1}^{1} \bar{b}_k\,(u\cdot p)^k$$

$$b_1 = \bar{b}_1 = \chi_b - \frac{\beta}{3} \qquad \text{and,} \qquad b_{-1} = \bar{b}_{-1} = \frac{m^2\beta}{3},$$

$$\bar{b}_0 = b_0 + 2\,\chi_a.$$